AN EXAMPLE OF THE DERIVED GEOMETRICAL SATAKE CORRESPONDENCE OVER INTEGERS

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Abstract. Let $G^\vee$ be a complex simple algebraic group. We describe certain morphisms of $G^\vee(\mathcal{O})$-equivariant complexes of sheaves on the affine Grassmannian Gr of $G^\vee$ in terms of certain morphisms of G-equivariant coherent sheaves on $\mathfrak{g}$, where G is the Langlands dual group of $G^\vee$ and $\mathfrak{g}$ is its Lie algebra. This can be regarded as an example of the derived Satake correspondence.

Introduction

Let $G^\vee$ be a complex reductive group, and Gr be its affine Grassmannian. Let k be a commutative unital noetherian ring of finite global dimension. The geometrical Satake correspondence (cf. [G, MV]) asserts that the Satake category $\mathcal{P}_k$ of $G^\vee(\mathcal{O})$-equivariant perverse sheaves with k-coefficients on Gr is equivalent as a tensor category to the category of representations of the Langlands dual group G_k defined over k. The Satake category arises as the heart of the natural t-structure on a triangulated category, the equivariant derived category $D_{G^\vee(\mathcal{O})}(\mathrm{Gr})$. And it is a natural question, in particular asked by Drinfeld, to describe this so-called derived Satake category in terms of G_k[1]. When the coefficients k is a field of characteristic zero, a solution is given in (cf. [ABG, BeF]). Namely,

$$DS : D_{G^\vee(\mathcal{O})}(\mathrm{Gr}) \cong D^G_{perf}(S\mathfrak{g}^*). \tag{0.1}$$

Here, $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G_k, and $D^G_{perf}(S\mathfrak{g}^*)$ is a "differential graded version" of the derived category of G-equivariant coherent sheaves on $\mathfrak{g}$. The compatibility of the equivalence (0.1) with the geometrical Satake isomorphism is as follows. Let $\mathcal{F}$ be a $G^\vee(\mathcal{O})$-equivariant perverse sheaf on Gr, and $V = H^*(\mathrm{Gr}, \mathcal{F})$ be the representation of G_k under the geometrical correspondence. Then $DS(\mathcal{F}) \cong V \otimes S\mathfrak{g}^*$, where the grading(resp. G-action) on $V \otimes S\mathfrak{g}^*$ is the product grading (resp. G-action).

Since the geometrical Satake correspondence holds for $k = \mathbb{Z}$, it is natural to expect that the equivalence (0.1) or at least the functor DS should extend to integers, after possibly inverting some small primes[2]. Let us be a little more precise. Let $\mathbb{Z}_S$ denote the ring of integers after inverting some small primes. Then one can attach every Chevalley group scheme G over $\mathbb{Z}_S$ its regular centralizer group scheme J (see [Ng] or §3.1 for its definition). According to a result of [YZ], the equivariant hypercohomology functor $H_{G^\vee(\mathcal{O})}$ gives a natural functor $D_{G^\vee(\mathcal{O})}(\mathrm{Gr}) \to D(J\text{−Mod})$. On

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[1] Indeed, the derived Satake category $D_{G^\vee(\mathcal{O})}(\mathrm{Gr})$ has more structures than barely a triangulated category, and Drinfeld asked to describe these finer structures.

[2] The author was informed that Dennis Gaitsgory and Jacob Lurie have made significant progress to describe the full structure of the derived Satake category with arbitrary coefficients.

the other hand, there is a tautological functor $D^G(S\mathfrak{g}^*) \to D(J\text{−Mod})$ (see (3.1)). One naturally expects that there is a functor $DS : D_{G^\vee(\mathcal{O})}(\mathrm{Gr}) \to D^G(S\mathfrak{g}^*)$ that lifts $H_{G^\vee(\mathcal{O})}$. Although such a lifting is not known to the author at the moment, the goal of this note is to compare some natural morphisms in $D_{G^\vee(\mathcal{O})}(\mathrm{Gr})$ with some natural morphisms in $D^G(S\mathfrak{g}^*)$ by identifying their images in $D(J\text{−Mod})$.

1. Main results and notations

1.1. We will assume that $G^\vee$ is simple and of adjoint type over $\mathbb{C}$. Let $\mathcal{O} = \mathbb{C}[[t]]$ and $F = \mathbb{C}((t))$. Then the affine Grassmannian $\mathrm{Gr} = G^\vee(F)/G^\vee(\mathcal{O})$ of $G^\vee$ is a union of projective varieties. To see this, we fix a Borel subgroup $B^\vee \subset G^\vee$ and a maximal torus $T^\vee \subset B^\vee$. Then each coweight λ of $T^\vee$ determines a point $t^\lambda \in T^\vee(F)$, and hence a point in Gr, which we still denote by t^λ. Let Gr^λ be the $G^\vee(\mathcal{O})$-orbit through t^λ. Each $G^\vee(\mathcal{O})$-orbit of Gr contains a unique point t^λ for some *dominant* coweight λ. We denote the closure of Gr^λ by $\overline{\mathrm{Gr}}^\lambda$. Then each $\overline{\mathrm{Gr}}^\lambda$ is a projective variety and Gr is their union.

We denote $i^\lambda : \mathrm{Gr}^\lambda \to \mathrm{Gr}$ to be the natural locally closed embedding. Then $i^\lambda_! := i^\lambda_!\underline{\mathbb{Z}}[\dim \mathrm{Gr}^\lambda]$ and $i^\lambda_* := i^\lambda_*\underline{\mathbb{Z}}[\dim \mathrm{Gr}^\lambda]$ are naturally objects in $D_{G^\vee(\mathcal{O})}(\mathrm{Gr}, \mathbb{Z})$. In addition, their degree zero perverse cohomology $I^\lambda_! := {}^p i^\lambda_!$ and $I^\lambda_* = {}^p i^\lambda_*$ are the standard and the costandard objects in $\mathcal{P}_\mathbb{Z}$. There is a natural sequence of maps $i^\lambda_! \to I^\lambda_! \to I^\lambda_* \to i^\lambda_*$, which gives

$$H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i^\lambda_!) \to H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I^\lambda_!) \to H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I^\lambda_*) \to H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i^\lambda_*). \tag{1.1}$$

Let us recall the description of the integral homology of Gr in terms of $G_\mathbb{Z}$, the Langlands dual group of $G^\vee$ over $\mathbb{Z}$ (see below), following [YZ] (the description of the rational cohomology of Gr in terms of $G_\mathbb{Q}$ was obtained by Ginzburg (cf. [G])). The $G^\vee(\mathcal{O})$-equivariant homology of Gr is a commutative and cocommutative Hopf algebra over $H(\mathbb{B}G^\vee, \mathbb{Z})$, and therefore $J = \mathrm{Spec}H^{G^\vee(\mathcal{O})}_*(\mathrm{Gr}, \mathbb{Z})$ is a commutative group scheme over $\mathrm{Spec}H(\mathbb{B}G^\vee, \mathbb{Z})$.

Let S be the multiplicative set generated by the bad primes of $G^\vee$ (i.e., those dividing the coefficients of the highest root in terms of a linear combination of simple roots) and those dividing $n+1$ if $G^\vee$ is of type A_n. Let $\mathbb{Z}_S$ be the localization of $\mathbb{Z}$ by S. Then according to [YZ], J is canonically isomorphic to the regular centralizer of $G_{\mathbb{Z}_S}$ over $\mathbb{Z}_S$ (see §3.1 for the review of the regular centralizer). Observe that for any $\mathcal{F} \in D_{G^\vee(\mathcal{O})}(\mathrm{Gr}, \mathbb{Z}_S)$, the hypercohomology $H^*_{G^\vee(\mathcal{O})}(\mathrm{Gr}, \mathcal{F})$ is a module over J. The goal of the note is to describe the sequence (1.1) as J-modules.

1.2. For brevity, we will suppress the subscript $\mathbb{Z}_S$ so that we write G for $G_{\mathbb{Z}_S}$, which is a simply-connected Chevalley group over $\mathbb{Z}_S$. Let $T \subset B$ be the maximal torus and the Borel subgroup of G dual to $T^\vee \subset B^\vee$. Let $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ be their Lie algebras. Let λ be a dominant weight of G w.r.t. B, and $P_\lambda \supset B$ be the standard parabolic subgroup of G corresponding to λ. That is, the Weyl group of P_λ coincides with the stablizer of λ in the Weyl group of G. Let $\mathfrak{p}_\lambda$ be the Lie algebra of P_λ. Let $\mathcal{P}_\lambda$ the moduli scheme of parabolic subgroups of G conjugate to P_λ. There is an ample invertible sheaf $\mathcal{O}(\lambda)$ on $\mathcal{P}_\lambda$, such that $\Gamma(\mathcal{P}_\lambda, \mathcal{O}(\lambda))^*$ is isomorphic the Weyl module W^λ of G of highest weight λ. Then $\Gamma(\mathcal{P}_\lambda, \mathcal{O}(\lambda))$ is isomorphic to the Schur module $S^{-w_0(\lambda)}$ of G, where w_0 is the longest element in the Weyl group W of G.

Next consider the partial Grothendieck alteration

$$\tilde{\mathfrak{g}}_\lambda = G \times^{P_\lambda} \mathfrak{p}_\lambda.$$

It embeds into $\mathcal{P}_\lambda \times \mathfrak{g}$ via

$$G \times^{P_\lambda} \mathfrak{p}_\lambda \to G \times^{P_\lambda} \mathfrak{g} \cong G/P_\lambda \times \mathfrak{g}. \tag{1.2}$$

Let us use the following notation. If $f : X \to \mathcal{P}_\lambda$ is a morphism, and $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{F} \otimes f^*\mathcal{O}(\lambda)$ is denoted by $\mathcal{F}(\lambda)$. Let $p : \mathcal{P}_\lambda \times \mathfrak{g} \to \mathfrak{g}$ be the projection to the second factor. We thus obtain a map of G-equivariant coherent sheaves over $\mathfrak{g}$,

$$p_*\mathcal{O}_{\mathcal{P}_\lambda \times \mathfrak{g}}(\lambda) \to p_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda). \tag{1.3}$$

According to (3.1), there is a functor from the category of G-equivariant sheaves on $\mathfrak{g}$ to the category of J-module. Let us denote the J-module corresponding to $p_*\mathcal{O}_{\mathcal{P}_\lambda \times \mathfrak{g}}(\lambda)$ by $\mathcal{S}^{-w_0(\lambda)}$ and to $p_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda)$ by $\mathcal{L}_*^{-w_0(\lambda)}$. Let $\mathcal{W}^\lambda$ (resp. $\mathcal{L}_!^\lambda$) be the dual of $\mathcal{S}^{-w_0(\lambda)}$ (resp. $\mathcal{L}_*^{-w_0(\lambda)}$) as J-modules. We thus obtain a sequence of J-module maps

$$\mathcal{L}_!^\lambda \to \mathcal{W}^\lambda \to \mathcal{S}^\lambda \to \mathcal{L}_*^\lambda, \tag{1.4}$$

where $\mathcal{W}^\lambda \to \mathcal{S}^\lambda$ essentially comes from $W^\lambda \to S^\lambda$.

The main result of this note is

Theorem 1.1. *Over $\mathbb{Z}_S$, the sequence of maps* (1.1) *is canonically identified with* (1.4) *as J-modules.*

This theorem has the following specialization. Let e be a regular nilpotent element of $\mathfrak{g}$. Then $(\mathcal{P}_\lambda)_e := e \times_{\mathfrak{g}} \tilde{\mathfrak{g}}_\lambda$ is called the (thick) Springer fiber of e, which is isomorphic to the scheme of zero locus of the vector field on $\mathcal{P}_\lambda$ determined by e.

Corollary 1.2. *Under the isomorphism of G-modules,*

$$\Gamma(\mathcal{P}_\lambda, \mathcal{O}(\lambda)) \cong S^{-w_0(\lambda)} \cong H(\mathrm{Gr}, I_*^{-w_0(\lambda)}),$$

the map $\Gamma(\mathcal{P}_\lambda, \mathcal{O}(\lambda)) \to \Gamma((\mathcal{P}_\lambda)_e, \mathcal{O}(\lambda))$ *corresponds to* $H(\mathrm{Gr}, I_*^{-w_0(\lambda)}) \to H(\mathrm{Gr}^{-w_0(\lambda)})$.

This corollary was originally conjectured by Ginzburg (cf. [Bez, Remark 10]).

1.3. The main ingredient to prove Theorem 1.1 is

Theorem 1.3. *Over $\mathbb{Z}_S$, the natural map* (1.3) *is surjective.*

The proof of Theorem 1.3, in turn, relies on the following result. Let p be a very good prime of G (i.e., those do not belong to S). Then $G \otimes \bar{\mathbb{F}}_p$ is the Chevalley group over $\bar{\mathbb{F}}_p$, and we have the corresponding partial Grothendieck alteration for $G \otimes \bar{\mathbb{F}}_p$, which is just the base change of the partial Grothendieck alteration (1.2) of G to $\bar{\mathbb{F}}_p$. We have

Theorem 1.4. *Let $p > 0$ be a very good prime of G. Then over $\bar{\mathbb{F}}_p$, the natural embedding $\tilde{\mathfrak{g}}_\lambda \to \mathcal{P}_\lambda \times \mathfrak{g}$ admits a compatibly Frobenius splitting.*

The basic facts about the Frobenius splitting will be recalled in §2.1.

1.4. **Plan of the paper.** In §2, after reviewing the basic facts of the Frobenius splitting, we will prove Theorem 1.4 and Theorem 1.3. In §3, after reviewing the regular centralizer group scheme and the equivariant homology of the affine Grassmannian, we will prove Theorem 1.1.

1.5. **Further conventions and notations.** As mentioned above, $(G^\vee, B^\vee, T^\vee)$ will denote a complex simple group of adjoint type together with a Borel subgroup and a maximal torus contained in this Borel subgroups. Let (G, B, T) denote the Langlands dual of $(G^\vee, B^\vee, T^\vee)$ over $\mathbb{Z}_S$ and $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{t}$ be their Lie algebras. For a weight ν of T and a base field k, the 1-dimensional representation of T_k (and therefore of B_k) corresponding to ν will be denoted as k^ν. The full flag variety of G is denoted by $\mathcal{B} = G/B$. The full Grothendieck alteration is denoted by $\tilde{\mathfrak{g}} = G \times^B \mathfrak{b}$.

All $G^\vee(\mathcal{O})$-equivariant complexes of sheaves on Gr are taken $\mathbb{Z}_S$-coefficients. The (co)homology $H^*(-)$ and $H_*(-)$ are also taken $\mathbb{Z}_S$.

For an affine scheme A, we will use $\mathcal{O}_A$ to denote the ring of functions on A. More generally, if $\mathcal{F}$ is a quasi-coherent sheaf on A, we will denote the space of its global sections also by $\mathcal{F}$.

If H is an affine group scheme over some base, and X is an affine H-scheme over the base, then we will denote $X/\!\!/H = \operatorname{Spec} \mathcal{O}_X^H$ to be the GIT quotient.

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2. Proof of Theorem 1.4, Theorem 1.3

In this section, we prove Theorem 1.4 and its Corollary 1.3.

2.1. **Generalities on the Frobenius splitting.** In this subsection, k will denote an algebraically closed field of characteristic $p > 0$. Let us briefly recall the general setting.

Let X be a scheme defined over k. The Frobenius twist X' of X is the base change of X along the absolutely Frobenius morphism of Spec$k$. Then there is a relative Frobenius morphism $Fr : X \to X'$.

Recall that (cf. [MR]) X is called Frobenius split if the $\mathcal{O}_{X'}$-module map $\mathcal{O}_{X'} \to Fr_*\mathcal{O}_X$ admits a splitting map (i.e. an $\mathcal{O}_{X'}$-module map $\varphi : Fr_*\mathcal{O}_X \to \mathcal{O}_{X'}$ satisfying $\varphi(1) = 1$). Such a splitting map is called a Frobenius splitting. If $i : Y \to X$ is a closed embedding defined by an ideal sheaf $\mathcal{I}_Y$, then Y is called compatibly split in X if there is a splitting $\varphi : Fr_*\mathcal{O}_X \to \mathcal{O}_{X'}$ which maps $Fr_*\mathcal{I}_Y$ to $\mathcal{I}_{Y'}$. In this case, it induces a split $\bar{\varphi} : Fr_*\mathcal{O}_Y \to \mathcal{O}_{Y'}$.

Assume that X is a smooth scheme over k. Then the Grothendieck duality implies that $\mathcal{H}om(Fr_*\mathcal{O}_X, \mathcal{O}_{X'}) \cong Fr_*\omega_X^{1-p}$, where ω_X is the canonical sheaf of X. Therefore, we will call a section of $Fr_*\omega_X^{1-p}$ a *splitting section* if it gives rise to a splitting of X via the above isomorphism. Furthermore, using the Cartier operator, the above isomorphism was written down explicitly in [MR]. We recall it in a form we need here.

Lemma 2.1. *Let $X = \operatorname{Spec} k[x_1, \dots, x_n]$. We will identify the natural map $\mathcal{O}_{X'} \to Fr_*\mathcal{O}_X$ as $k[x_1^p, \dots, x_n^p] \to k[x_1, \dots, x_n]$ and $Fr_*\omega_X^{1-p}$ as $k[x_1, \dots, x_n](dx_1 \wedge \cdots \wedge dx_n)^{1-p}$. We will adopt the multiple-index notation, so that $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$*

for $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ *and* $dx = dx_1 \wedge \dots \wedge dx_n$. *Furthermore, if* m *is an integer, we denote* $\underline{m} = (m, \dots, m) \in \mathbb{Z}^n$. *Then the isomorphism* $Fr_*\omega_X^{1-p} \cong \mathcal{H}om(Fr_*\mathcal{O}_X, \mathcal{O}_{X'})$ *is given by*

$$x^a(dx)^{1-p}(x^b) = \begin{cases} x^{\frac{a+b+\underline{1}-\underline{p}}{p}} & p|(a+b+\underline{1}-\underline{p}) \\ 0 & \text{otherwise.} \end{cases}$$

The same formula holds for $X = \mathrm{Spec}k[[x_1, \dots, x_n]]$.

We need another lemma in the sequel. The proof is easy and is left to the readers.

Lemma 2.2. *Let* X *be a scheme of finite type over* k *and* $Y \subset X$ *be a closed subscheme of* X. *Let* $x \in Y$ *be a closed point. Since* $Fr : X \to X'$ *is a homeomorphism, we can also regard* x *as a point in* X'. *Let* $\hat{\mathcal{O}}_{X,x}$ *(resp.* $\hat{\mathcal{O}}_{X',x}$*, resp.* $\hat{\mathcal{I}}_{Y,x}$*, resp.* $\hat{\mathcal{I}}_{Y',x}$*) be the completion of* $\mathcal{O}_X$ *(resp.* $\mathcal{O}_{X'}$*, resp.* $\mathcal{I}_Y$*, resp.* $\mathcal{I}_{Y'}$*) at* x. *For any* $\mathcal{O}_{X'}$*-module map* $f : Fr_*\mathcal{O}_X \to \mathcal{O}_{X'}$*, let* $f_x : \hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{X',x}$ *be the induced* $\hat{\mathcal{O}}_{X',x}$*-module map. Then: (i)* f *is a splitting map if and only if* f_x *splits the natural map* $\hat{\mathcal{O}}_{X',x} \to \hat{\mathcal{O}}_{X,x}$*.; and (ii)* $f(Fr_*\mathcal{I}_Y) \subset \mathcal{I}_{Y'}$ *if and only if* $f_x(\hat{\mathcal{I}}_{Y,x}) \subset \hat{\mathcal{I}}_{Y',x}$.

2.2. **Proof of Theorem 1.4.** In this subsection, we assume that G is simple and simply-connected over an algebraically closed field of characteristic $p > 0$. Since all the schemes in the subsection are in fact defined over $\mathbb{F}_p$, we will not distinguish them from their Frobenius twists.

Let $P \subset G$ be a parabolic subgroup and $\mathfrak{p} \subset \mathfrak{g}$ be its parabolic subalgebra, and $\mathcal{P} = G/P$ be the variety of parabolic subgroups of G that are conjugate to P. Let $\tilde{\mathfrak{g}}_{\mathcal{P}} := G \times^P \mathfrak{p}$ be the partial Grothendieck alteration. It embeds into $G \times^P \mathfrak{g} \cong \mathcal{P} \times \mathfrak{g}$ as a closed subscheme. If the characteristic p is very good for G, then $\mathcal{P}$ can be also regarded as the variety of parabolic subalgebras of $\mathfrak{g}$ that are conjugate to $\mathfrak{p}$ (since in this case the normalizer of $\mathfrak{p}$ in G is P), and $\tilde{\mathfrak{g}}_{\mathcal{P}}$ can be regarded as the variety of pairs $(\mathfrak{p}', \xi)$, where $\mathfrak{p}'$ is a parabolic subalgebra of $\mathfrak{g}$ conjugate to $\mathfrak{p}$ and $\xi \in \mathfrak{p}'$. We will prove the following theorem.

Theorem 2.3. *Assume that the characteristic of* k *is* $p > 0$ *and* p *is very good for* G. *Then the closed embedding* $\tilde{\mathfrak{g}}_{\mathcal{P}} \to \mathcal{P} \times \mathfrak{g}$ *admits a compatibly Frobenius splitting.*

Proof. The proof is divided into several steps.

(i) Consider the natural projection $\mathrm{pr} : \mathcal{B} \times \mathfrak{g} \to \mathcal{P} \times \mathfrak{g}$. It is a proper morphism satisfying $\mathrm{pr}_*\mathcal{O}_{\mathcal{B}\times\mathfrak{g}} = \mathcal{O}_{\mathcal{P}\times\mathfrak{g}}$, which maps $\tilde{\mathfrak{g}}_{\mathcal{B}}$ onto $\tilde{\mathfrak{g}}_{\mathcal{P}}$. Therefore, according to [MR, Proposition 4], it is enough to prove the theorem for $\mathcal{P} = \mathcal{B}$.

(ii) Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$ so that $\mathcal{O}_{\mathfrak{g}} = S\mathfrak{g}^*$ is the symmetric algebra over $\mathfrak{g}^*$. It decomposes as G-modules according to the natural grading $S\mathfrak{g}^* = \sum_n S^n\mathfrak{g}^*$.

Let $\pi : \mathcal{B} \times \mathfrak{g} \to \mathcal{B}$ be the projection. Then $\omega_{\mathcal{B}\times\mathfrak{g}} \cong \pi^*\omega_{\mathcal{B}} \cong \pi^*\mathcal{O}_{\mathcal{B}}(-2\rho)$. Using the isomorphism $\mathcal{B} \times \mathfrak{g} \cong G \times^B \mathfrak{g}$, one can identify

$$\begin{aligned} &\Gamma(\mathcal{B} \times \mathfrak{g}, Fr_*\omega_{\mathcal{B}\times\mathfrak{g}}^{1-p}) \cong \Gamma(\mathcal{B}, \pi_*\pi^*\omega_{\mathcal{B}}^{1-p}) \\ &\cong (\mathcal{O}_G \otimes (\mathcal{O}_{\mathfrak{g}} \otimes k^{-2(p-1)\rho}))^B = \bigoplus_n (\mathcal{O}_G \otimes (S^n\mathfrak{g}^* \otimes k^{-2(p-1)\rho}))^B, \end{aligned}$$

where $\mathcal{O}_{\mathfrak{g}} \otimes k^{-2(p-1)\rho}$ is regarded as a B-module. Let $d = (p-1)\dim\mathfrak{g}$. We will use the homogeneous piece $(\mathcal{O}_G \otimes (S^d\mathfrak{g}^* \otimes k^{-2(p-1)\rho}))^B$.

Let us define a natural nonzero G-module homomorphism

$$\varepsilon : S^d\mathfrak{g}^* \to k$$

as follows. Let I be the ideal of $\mathcal{O}_\mathfrak{g}$ generated by $\{v^p,\ v \in \mathfrak{g}^*\}$. This is a G-submodule of $\mathcal{O}_\mathfrak{g}$. Then one has the short exact sequence of G-modules

$$0 \to S^d\mathfrak{g}^* \cap I \to S^d\mathfrak{g}^* \xrightarrow{\varepsilon} k \to 0. \tag{2.1}$$

Such a G-module homomorphism gives a B-module homomorphism, still denoted by ε,

$$\varepsilon : S^d\mathfrak{g}^* \otimes k^{-2(p-1)\rho} \to k^{-2(p-1)\rho}.$$

Therefore, we obtain the following map

$$\begin{aligned} \operatorname{ind}(\varepsilon) :&(\mathcal{O}_G \otimes (S^d\mathfrak{g}^* \otimes k^{-2(p-1)\rho}))^B \\ &\to (\mathcal{O}_G \otimes k^{-2(p-1)\rho})^B = \Gamma(\mathcal{B}, Fr_*\omega_\mathcal{B}^{1-p}). \end{aligned}$$

Lemma 2.4. *A section $\sigma \in (\mathcal{O}_G \otimes (S^d\mathfrak{g}^* \otimes k^{-2(p-1)\rho}))^B \subset \Gamma(\mathcal{B}\times\mathfrak{g}, Fr_*\omega_{\mathcal{B}\times\mathfrak{g}}^{1-p})$ is a splitting section of $\mathcal{B}\times\mathfrak{g}$ if and only if $\operatorname{ind}(\varepsilon)(\sigma)$ is a splitting section of $\mathcal{B}$.*

Proof. The method used here is similar to [KLT]. Let U_- be the unipotent radical of B_-, which is the Borel subgroup of G opposite to B. Let $U_-\cdot[1]$ be the big cell of $\mathcal{B}$, where $[1]$ denotes our chosen Borel subgroup $B \subset G$. We choose a system of homogeneous coordinates $\{x_\alpha, \alpha \in \Delta_+\}$ of U_-, i.e. $U_- = \operatorname{Spec} k[x_\alpha, \alpha \in \Delta_+]$, where x_α is a T-weight function of U_- of weight α. Let us also choose a system of homogeneous coordinates $\{y_i \in \mathfrak{g}^*, 1 \le i \le \dim\mathfrak{g}\}$ for $\mathfrak{g}$.

An element $\sigma \in (\mathcal{O}_G \otimes (S^d\mathfrak{g}^* \otimes k^{-2(p-1)\rho}))^B \subset \Gamma(\mathcal{B}\times\mathfrak{g}, Fr_*\omega_{\mathcal{B}\times\mathfrak{g}}^{1-p})$ restricts over $U_-\cdot[1]$ to an element of the form

$$\operatorname{res}(\sigma) = f(dx\wedge dy)^{1-p} \in Fr_*\omega_{U_-\cdot[1]\times\mathfrak{g}}^{1-p} \cong \mathcal{O}_{U_-}\otimes S\mathfrak{g}^*\otimes k^{-2(p-1)\rho}, \quad f \in \mathcal{O}_{U_-}\otimes S^d\mathfrak{g}^*.$$

According to Lemma 2.1, this is a splitting section of $U_-\cdot[1]\times\mathfrak{g}$ if and only if the coefficient of the monomial $x^{\underline{p-1}}y^{\underline{p-1}}$ appearing in f is not zero and the coefficients of the monomials $x^{\underline{p-1}+pa}y^{\underline{p-1}+pb}((a,b) \neq 0 \in \mathbb{Z}_{\ge 0}^{\dim U_-}\times\mathbb{Z}_{\ge 0}^{\dim\mathfrak{g}})$ appearing in f are zero. Since $\sigma \in \mathcal{O}_{U_-}\otimes S^d\mathfrak{g}^*$, no monomials of the form $x^a y^{\underline{p-1}+pb}(b \neq 0 \in \mathbb{Z}_{\ge 0}^{\dim\mathfrak{g}})$ appear in f (the degree of $y^{\underline{p-1}+pb}(b \neq 0 \in \mathbb{Z}_{\ge 0}^{\dim\mathfrak{g}})$ is greater than d).

On the other hand, the section $\operatorname{ind}(\varepsilon)(\sigma) \in (\mathcal{O}_G\otimes k^{-2(p-1)\rho})^B = \Gamma(\mathcal{B}, Fr_*\omega_\mathcal{B}^{1-p})$ restricts over $U_-\cdot[1]$ to an element of the form

$$\operatorname{res}(\operatorname{ind}(\varepsilon)(\sigma)) = g(dx)^{1-p} \in Fr_*\omega_{U_-\cdot[1]}^{1-p} \cong \mathcal{O}_{U_-}\otimes k^{-2(p-1)\rho}, \quad g \in \mathcal{O}_{U_-}.$$

Again by Lemma 2.1, this is a splitting section of $U_-\cdot[1]$ if and only if the coefficient of the monomial $x^{\underline{p-1}}$ appearing in g is not zero and the coefficients of the monomials $x^{\underline{p-1}+pa}, (a \neq 0 \in \mathbb{Z}_{\ge 0}^{\dim U_-})$ appearing in g are zero.

Now let $\sigma \in (\mathcal{O}_G \otimes (S^d\mathfrak{g}^* \otimes k^{-2(p-1)\rho}))^B$. Recall the notations

$$\operatorname{res}(\sigma) = f(dx\wedge dy)^{1-p}, f \in \mathcal{O}_{U_-}\otimes S^d\mathfrak{g}^*, \qquad \operatorname{res}(\operatorname{ind}(\varepsilon)(\sigma)) = g(dx)^{1-p}, g \in \mathcal{O}_{U_-}.$$

By the following commutative diagram

$$\begin{tikzcd}
(\mathcal{O}_G \otimes (S^d\mathfrak{g}^* \otimes k^{-2(p-1)\rho}))^B \arrow[r, "\mathrm{ind}(\varepsilon)"] \arrow[d, "\mathrm{res}"'] & (\mathcal{O}_G \otimes k^{-2(p-1)\rho})^B \arrow[d, "\mathrm{res}"'] \\
\mathcal{O}_{U_-} \otimes (S^d\mathfrak{g}^* \otimes k^{-2(p-1)\rho}) \arrow[r, "\mathrm{id}\otimes\varepsilon"] & \mathcal{O}_{U_-} \otimes k^{-2(p-1)\rho}
\end{tikzcd}$$

and the definition of ε (see (2.1)), if we write $f = f_1 y^{\underline{p-1}} +$ (other terms) for some $f_1 \in \mathcal{O}_{U_-}$, then $g = f_1$.

Therefore, if $\mathrm{ind}(\varepsilon)(\sigma)$ is a splitting section of $\mathcal{B}$, then the monomial $x^{\underline{p-1}+pa}(a \in \mathbb{Z}_{\geq 0}^{\dim U_-})$ appears in $g = f_1$ if and only if $a = 0$, which implies the monomial $x^{\underline{p-1}+pa} y^{\underline{p-1}+pb}((a,b) \in \mathbb{Z}_{\geq 0}^{\dim U_-} \times \mathbb{Z}_{\geq 0}^{\dim \mathfrak{g}})$ appears in f if and only if $(a,b) = 0$. This in turn implies that σ is a splitting section of $\mathcal{B} \times \mathfrak{g}$. By the same argument, the converse holds and the lemma is proven. □

(iii) Assume that $\sigma \in (\mathcal{O}_G \otimes (S\mathfrak{g}^* \otimes k^{-2(p-1)\rho}))^B \cong \mathrm{Hom}(Fr_*\mathcal{O}_{\mathcal{B}\times\mathfrak{g}}, \mathcal{O}_{\mathcal{B}\times\mathfrak{g}})$. Let us see when $\sigma(Fr_*\mathcal{I}_{\tilde{\mathfrak{g}}}) \subset \mathcal{I}_{\tilde{\mathfrak{g}}}$, so that it induces a map $\sigma : Fr_*\mathcal{O}_{\tilde{\mathfrak{g}}} \to \mathcal{O}_{\tilde{\mathfrak{g}}}$, where $\mathcal{I}_{\tilde{\mathfrak{g}}}$ is the sheaf of ideals defining $\tilde{\mathfrak{g}} \subset \mathcal{B} \times \mathfrak{g}$. Let $\mathcal{I}_{\mathfrak{b}} \subset \mathcal{O}_{\mathfrak{g}}$ denote the ideal defining $\mathfrak{b} \subset \mathfrak{g}$. We define a B-submodule $J \subset \mathcal{O}_{\mathfrak{g}}$ as follows. Observe there is a unique up to scalar $\mathcal{O}_{\mathfrak{g}}$-module isomorphism $\mathcal{O}_{\mathfrak{g}} \cong \omega_{\mathfrak{g}}$. By Composing it with the isomorphism in Lemma 2.1, we obtain an isomorphism (up to scalar) $Fr_*\mathcal{O}_{\mathfrak{g}} \cong \mathrm{Hom}(Fr_*\mathcal{O}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})$. We define

$$J = \{\delta : Fr_*\mathcal{O}_{\mathfrak{g}} \to \mathcal{O}_{\mathfrak{g}}, \quad \delta(Fr_*\mathcal{I}_{\mathfrak{b}}) \subset \mathcal{I}_{\mathfrak{b}}\}. \tag{2.2}$$

Here is a more concrete description of $J \subset S\mathfrak{g}^* = Fr_*\mathcal{O}_{\mathfrak{g}}$, from which the B-module structure of J is clear. Fixing the maximal torus T, we have the decomposition $\mathfrak{g} = \mathfrak{u}_- + \mathfrak{b}$. But as B-modules, we only have

$$0 \to \mathfrak{u}_-^* \to \mathfrak{g}^* \to \mathfrak{b}^* \to 0.$$

Let $J_1 = \mathrm{Im}(S^{(p-1)\dim \mathfrak{u}_-}\mathfrak{u}_-^* \otimes S\mathfrak{g}^* \to S\mathfrak{g}^*)$. This is a B-submodule of $S\mathfrak{g}^*$. On the other hand, let J_2 be the ideal of $S\mathfrak{g}^*$ generated by $\{v^p,\ v \in \mathfrak{u}_-^*\}$. which is also a B-submodule of $S^d\mathfrak{g}^*$. Then $J = J_1 + J_2$ by Lemma 2.1. Observe that $J = \oplus_n J^n$, where $J^n = J \cap S^n\mathfrak{g}^*$.

Lemma 2.5. *A section $\sigma \in (\mathcal{O}_G \otimes (S\mathfrak{g}^* \otimes k^{-2(p-1)\rho}))^B \cong \mathrm{Hom}(Fr_*\mathcal{O}_{\mathcal{B}\times\mathfrak{g}}, \mathcal{O}_{\mathcal{B}\times\mathfrak{g}})$ maps $Fr_*\mathcal{I}_{\tilde{\mathfrak{g}}}$ to $\mathcal{I}_{\tilde{\mathfrak{g}}}$ if and only if $\sigma \in (\mathcal{O}_G \otimes (J \otimes k^{-2(p-1)\rho}))^B \subset (\mathcal{O}_G \otimes (S\mathfrak{g}^* \otimes k^{-2(p-1)\rho}))^B$.*

Proof. It is enough to see when $\sigma(Fr_*\mathcal{I}_{\tilde{\mathfrak{g}}}) \subset \mathcal{I}_{\tilde{\mathfrak{g}}}$ over $U_- \cdot [1]$. Then the lemma follows the first description of J. □

(iv) From previous two lemmas, to finish the proof of the theorem, it is enough to construct a section $\sigma \in (\mathcal{O}_G \otimes (J^d \otimes k^{-2(p-1)\rho}))^B$ such that $\mathrm{ind}(\varepsilon)(\sigma)$ gives a splitting of $\mathcal{B}$.

Let $\mathrm{St} := W^{(p-1)\rho} = S^{(p-1)\rho}$ be the first Steinberg module of G, which is irreducible and selfdual. Here ρ is the sum of fundamental weights of T. Let us fix a G-invariant non-degenerate bilinear form $(\cdot,\cdot)$ on St. The theorem then would follow if we could construct a B-module homomorphism

$$\gamma : \mathrm{St} \otimes \mathrm{St} \to J^d \otimes k^{-2(p-1)\rho} \quad \text{s.t.} \quad \varepsilon \circ \gamma \neq 0 : \mathrm{St} \otimes \mathrm{St} \to k^{-2(p-1)\rho}. \tag{2.3}$$

This is because then we would have the following nonzero G-module maps

$$\mathrm{St}\otimes\mathrm{St}\overset{\mathrm{ind}(\gamma)}{\to}(\mathcal{O}_G\otimes(J^d\otimes k^{-2(p-1)\rho}))^B\overset{\mathrm{ind}(\varepsilon)}{\to}(\mathcal{O}_G\otimes k^{-2(p-1)\rho})^B,$$

and according to the main theorem of [LT], any $\sigma=\mathrm{ind}(\gamma)(v\otimes w)$ for $v\otimes w\in\mathrm{St}\otimes\mathrm{St},\ (v,w)\neq 0$ would satisfy our purpose.

(v) It remains to construct a B-module homomorphism (2.3). However, let us first define a B-module homomorphism

$$\gamma_0:\mathrm{St}\otimes\mathrm{St}\otimes k^{2(p-1)\rho}\to\mathcal{O}_G$$

by the following formula: let v_+ (resp. v_-) be a nonzero highest (resp. lowest) weight vector in St, then

$$\gamma_0(v\otimes w\otimes v_+\otimes v_+)(g)=(v,gw)(v_+,gv_+),\quad v\otimes w\in\mathrm{St}\otimes\mathrm{St},g\in G.$$

Since ω_G is trivial and $\Gamma(G,\mathcal{O}^*)=k^*$, there is a unique (up to scalar) $\mathcal{O}_G$-module isomorphism $i:\mathcal{O}_G\cong\omega_G$. Thus, we obtain an isomorphism $\mathcal{O}_G\to\omega_G^{1-p}, f\mapsto fi(1)^{1-p}$, and by Lemma 2.1, we obtain a map, still denoted by γ_0

$$\gamma_0:\mathrm{St}\otimes\mathrm{St}\otimes k^{2(p-1)\rho}\to Fr_*\omega_G^{1-p}\cong\mathrm{Hom}(Fr_*\mathcal{O}_G,\mathcal{O}_G).$$

The main properties of γ_0 is summarized in the following lemma. Let $\mathcal{I}_B$ be the ideal defining $B\subset G$.

Lemma 2.6. *For any $\sigma\in\mathrm{St}\otimes\mathrm{St}\otimes k^{2(p-1)\rho}$, $\gamma_0(\sigma)(Fr_*\mathcal{I}_B)\subset\mathcal{I}_B$. Furthermore, $\gamma_0(v_-\otimes v_-\otimes v_+\otimes v_+)$ is a splitting section of G.*

Proof. Let $U_-B\subset G$ be the open subset of G. It is enough to prove the lemma over U_-B. Let us choose a system of homogenous coordinates $\{x_\alpha,\alpha\in\Delta_+\}$ (resp. $\{y_\alpha,-\alpha\in\Delta_+\}$) for U_- (resp. for U). And let t_i be the ith fundamental weight of T.

By construction, $\gamma_0(v\otimes w\otimes v_+\otimes v_+)=(v,gw)(v_+,gv_+)i(1)^{1-p}$. Since $i(1)$ is the unique (up to scalar) nonzero invariant differential form on G,

$$i(1)|_{U_-B}=(\prod_i t_i)^{-1}dxdydt\qquad\text{up to a scalar.}$$

On the other hand, it is clear that the function $g\mapsto(v_+,gv_+)$ and $g\mapsto(v_-,gv_-)$ restricted to U_-B has the form

$$f_1(x)(\prod t_i)^{p-1},f_1(x)\in k[x_\alpha,\alpha\in\Delta_+],$$
$$f_2(y)(\prod t_i)^{-(p-1)},f_2(y)\in k[y_\alpha,-\alpha\in\Delta_+].$$

Therefore,

$$\gamma_0(v\otimes w\otimes v_+\otimes v_+)|_{U_-B}=(v,gw)f_1(x)(\prod_i t_i)^{2(p-1)}(dxdydt)^{1-p},$$

and in particular

$$\gamma_0(v_-\otimes v_-\otimes v_+\otimes v_+)|_{U_-B}=f_2(y)f_1(x)(\prod_i t_i)^{(p-1)}(dxdydt)^{1-p}.$$

Since the T-weight of the function $f_1(x)$ is $2(p-1)\rho$, the monomial

$$x^a=\prod_{\alpha\in\Delta_+}x_\alpha^{a_\alpha},\quad a\in\mathbb{Z}_{\geq 0}^{\dim U_-}$$

appearing in $f_1(x)$ will be of the following two forms: either $\exists\alpha$, s.t. $a_\alpha \geq p$, or $a_\alpha = p-1$ for all $\alpha \in \Delta_+$. Therefore, according to Lemma 2.1, $\gamma_0(v\otimes w\otimes v_+\otimes v_+)(Fr_*\mathcal{I}_B|_{U_-B}) \subset \mathcal{I}_B|_{U_-B}$. The first statement of the lemma is proved. Next, it is shown in the proof of [KLT, Lemma 5] that the monomial $x^{\underline{p-1}}$ appears in $f_1(x)$ and none monomials of the form $x^{\underline{p-1}+pa}, a \neq 0 \in \mathbb{Z}_{\geq 0}^{\dim U_-}$ appear in $f_1(x)$. Similar results hold for $f_2(y)$. Then by Lemma 2.1 again, $\gamma_0(v_-\otimes v_-\otimes v_+\otimes v_+)|_{U_-B}$ gives a splitting of U_-B. The second statement of the lemma follows. ☐

(vi) Finally, let us see how γ_0 gives the desired map as in (2.3). Since the characteristic of p is assumed to be very good, there is a G-equivariant map $\varphi: G \to \mathfrak{g}$ which sends the unit $e \in G$ to the origin $0 \in \mathfrak{g}$, and induces the identity map $(D\varphi)_e = \mathrm{id}: T_eG \to T_0\mathfrak{g}$ (cf. [BR, 9.3.2-9.3.3]).

Lemma 2.7. *Such a map will necessarily map B to $\mathfrak{b}$.*

Proof. Let $2\check{\rho}$ be the sum of positive coroots of T. Then we have a morphism $2\check{\rho}: \mathbb{G}_m \to T$. We assume that $\mathbb{G}_m \subset \mathbb{A}^1$ naturally. Then $\mathfrak{b} \subset \mathfrak{g}$ has the following characterization. Let $x \in \mathfrak{g}(R)$ be any R-point of $\mathfrak{g}$, where R is a k-algebra. If the map $(\mathbb{G}_m)_R \to \mathfrak{g}_R$ defined by $t \mapsto \mathrm{Ad}(2\check{\rho}(t))x$ extends to $\mathbb{A}^1$, then $x \in \mathfrak{b}(R)$. Now, the conjugation of $2\check{\rho}(\mathbb{G}_m)$ on B extends to a morphism $\mathbb{A}^1 \times B \to B$. Therefore, by the G-equivariance of φ, φ will map B to $\mathfrak{b}$. ☐

Therefore, we obtain the following commutative diagram of B-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{\mathcal{I}}_{B,e} & \longrightarrow & \hat{\mathcal{O}}_{G,e} & \longrightarrow & \hat{\mathcal{O}}_{B,e} & \longrightarrow & 0 \\
 & & \cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow & & \\
0 & \longrightarrow & \hat{\mathcal{I}}_{\mathfrak{b},0} & \longrightarrow & \hat{\mathcal{O}}_{\mathfrak{g},0} & \longrightarrow & \hat{\mathcal{O}}_{\mathfrak{b},0} & \longrightarrow & 0.
\end{array}$$

Now by Lemma 2.2, we obtain a map of B-modules

$$\begin{aligned}
\mathrm{St}\otimes\mathrm{St}\otimes k^{2(p-1)\rho} \to &\mathrm{Hom}(Fr_*\mathcal{O}_G, \mathcal{O}_{G'}) \to \mathrm{Hom}(\hat{\mathcal{O}}_{G,e}, \hat{\mathcal{O}}_{G',e}) \\
\cong &\mathrm{Hom}(\hat{\mathcal{O}}_{\mathfrak{g},0}, \hat{\mathcal{O}}_{\mathfrak{g}',0}) \cong Fr_*\hat{\omega}_{\mathfrak{g},0}^{1-p}.
\end{aligned}$$

where $\hat{\omega}_{\mathfrak{g},0}$ denotes the completion of $\omega_{\mathfrak{g}}$ at 0. The unique up to scalar isomorphism $\omega_{\mathfrak{g}} \to \mathcal{O}_{\mathfrak{g}}$ induces a G-module isomorphism up to scalar $Fr_*\hat{\omega}_{\mathfrak{g},0}^{1-p} \to Fr_*\hat{\mathcal{O}}_{\mathfrak{g},0} \cong \prod_n S^n\mathfrak{g}^*$. Therefore, we obtain a map $\gamma: \mathrm{St}\otimes\mathrm{St}\otimes k^{2(p-1)\rho} \to S^d\mathfrak{g}^*$ by composition of above maps followed by the projection $\prod_n S^n\mathfrak{g}^* \to S^d\mathfrak{g}^*$. By Lemma 2.2, the first part of Lemma 2.6, and the definition of J (cf. (2.2)), we know the image of γ indeed lies in J^d. By Lemma 2.2, and the second part of Lemma 2.6, $\gamma(v_-\otimes v_-\otimes v_+\otimes v_+)$ is a splitting section of $\mathfrak{g}$, and therefore by Lemma 2.1, $(\varepsilon\circ\gamma)(v_-\otimes v_-\otimes v_+\otimes v_+) \neq 0$ (see (2.1) for the definition of ε). ☐

Remark 2.1. The existence of Frobenius splitting of $\tilde{\mathfrak{g}}$ is already proved (cf. [MvdK] Theorem 3.8 for type A case and cf. [KLT] Remark 1 for general case). However, to my knowlegdge, the existence of Frobenius splitting of $\tilde{\mathfrak{g}}_{\mathcal{P}}$ is not known before. Our approach is largely inspired by [KLT].

2.3. **Proof of Theorem 1.3.** Now we will deduce Theorem 1.3 as a consequence of the compatibly Frobenius splitting. First we still assume that everything is defined over $\bar{\mathbb{F}}_p$, with p being very good.

Proposition 2.8. *(i) The natural map*

$$\Gamma(\mathcal{P}_\lambda\times\mathfrak{g},\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}}(\lambda))\to\Gamma(\tilde{\mathfrak{g}}_\lambda,\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda))$$

is surjective.

(ii) We have

$$H^1(\mathcal{P}_\lambda\times\mathfrak{g},\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}}(\lambda))=H^1(\tilde{\mathfrak{g}}_\lambda,\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda))=0.$$

Proof. Since the closed embedding $\tilde{\mathfrak{g}}_\lambda\to\mathcal{P}_\lambda\times\mathfrak{g}$ is compatibly Frobenius splitting, by the standard argument (cf. [MR]), it is enough to prove that

$$\Gamma(\mathcal{P}_\lambda\times\mathfrak{g},\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}}(m\lambda))\to\Gamma(\tilde{\mathfrak{g}}_\lambda,\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(m\lambda)) \tag{2.4}$$

is surjective, and

$$H^1(\mathcal{P}_\lambda\times\mathfrak{g},\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}}(m\lambda))=H^1(\tilde{\mathfrak{g}}_\lambda,\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(m\lambda))=0, \tag{2.5}$$

for m sufficiently large. Since $\mathcal{P}_\lambda\times\mathfrak{g}$ is not projective, to prove this we need a little extra work.

Let $\pi:\mathcal{P}_\lambda\times\mathfrak{g}\to\mathcal{P}_\lambda$ be the projection to the first factor. Then the map (2.4) is identified with

$$\Gamma(\mathcal{P}_\lambda,\pi_*(\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}}(m\lambda)))\to\Gamma(\mathcal{P}_\lambda,\pi_*(\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(m\lambda))).$$

By the adjunction formula, the above map is the same as

$$\Gamma(\mathcal{P}_\lambda,\pi_*(\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}})(m\lambda))\to\Gamma(\mathcal{P}_\lambda,\pi_*(\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda})(m\lambda)),$$

induced from $S\mathfrak{g}^*\otimes\mathcal{O}_{\mathcal{P}_\lambda}\otimes\mathcal{O}(m\lambda)\to\pi_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}\otimes\mathcal{O}(m\lambda)$. It is clear that as $\mathcal{O}_{\mathcal{P}_\lambda}$-modules, there is an quasi-isomorphism $S\mathfrak{g}^*\otimes\Omega^\bullet_{\mathcal{P}_\lambda}\otimes\mathcal{O}(m\lambda)\to\pi_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}\otimes\mathcal{O}(m\lambda)$ (the Koszul resolution) extending the above morphism. Therefore, there is a spectral sequence converging to $H^{-p+q}(\mathcal{P}_\lambda,\pi_*(\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda})(m\lambda))$ with

$$E_1^{-p,q}=H^q(\mathcal{P}_\lambda,\Omega^p_{\mathcal{P}_\lambda}(m\lambda))\otimes S\mathfrak{g}^*$$

Now since $\mathcal{O}(\lambda)$ is ample, for m sufficiently large, $H^q(\mathcal{P}_\lambda,\Omega^p_{\mathcal{P}_\lambda}(m\lambda))=0$ for $q>0$. Therefore, $E_1^{0,0}\twoheadrightarrow E_\infty^{0,0}$. This proves the surjectivity of (2.4) for m sufficiently large. This argument also implies (2.5) at the same time. □

Now we begin to prove Theorem 1.3. Therefore, in the rest paragraph of this subsection, everything is defined over $\mathbb{Z}_S$. We want to show that

$$\Gamma(\mathcal{P}_\lambda\times\mathfrak{g},\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}}(\lambda))\to\Gamma(\tilde{\mathfrak{g}}_\lambda,\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda))$$

is surjective. It is enough to show that

$$\Gamma(\mathcal{P}_\lambda\times\mathfrak{g},\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}}(\lambda))\otimes\bar{\mathbb{F}}_p\to\Gamma(\tilde{\mathfrak{g}}_\lambda,\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda))\otimes\bar{\mathbb{F}}_p$$

is surjective for any $p\notin S$ (i.e. p is very good). By part (i) of Proposition 2.8, it is enough to proof that the canonical morphisms

$$\Gamma(\mathcal{P}_\lambda\times\mathfrak{g},\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}}(\lambda))\otimes\bar{\mathbb{F}}_p\to\Gamma((\mathcal{P}_\lambda\times\mathfrak{g})\otimes\bar{\mathbb{F}}_p,\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}}(\lambda))$$

and

$$\Gamma(\tilde{\mathfrak{g}}_\lambda,\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda))\otimes\bar{\mathbb{F}}_p\to\Gamma(\tilde{\mathfrak{g}}_\lambda\otimes\bar{\mathbb{F}}_p,\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda))$$

are isomorphisms. Let us prove the latter isomorphism, since the former is similar and even simpler. Again, let $\pi:\mathcal{P}_\lambda\times\mathfrak{g}\to\mathcal{P}_\lambda$ be the projection to the first factor. Then it is equivalent to prove that

$$\Gamma(\mathcal{P}_\lambda,\pi_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda))\otimes\bar{\mathbb{F}}_p\to\Gamma(\mathcal{P}_\lambda\otimes\bar{\mathbb{F}}_p,\pi_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda))$$

is surjective. Observe that $\pi_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda)$ is a direct sum of coherent sheaves on $\mathcal{P}_\lambda$, each of which is flat over $\mathbb{Z}_S$. By the standard base change theorem for cohomology, it is enough to show that $H^1(\mathcal{P}_\lambda\otimes\bar{\mathbb{F}}_p, \pi_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda)) = 0$ for every $p\notin S$, which is the content of part (ii) of Proposition 2.8. Therefore, Theorem 1.3 is proved.

2.4. **Flatness of $p_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}$ over $\mathfrak{g}^{reg}$.** Let $\mathfrak{g}^{reg}$ denote the open subscheme of $\mathfrak{g}$ consisting of regular elements. That is, for any $\mathbb{Z}_S$-scheme X, $\mathfrak{g}^{reg}(X)$ is the subset of $\mathfrak{g}(X)$ such that for any point $x\in X$, the composition $x\to X\to\mathfrak{g}$ is a regular element in $\mathfrak{g}\otimes\kappa(x)$, where $\kappa(x)$ is the residue field of x. It is clear that for any very good prime p, $\mathfrak{g}^{reg}\otimes\bar{\mathbb{F}}_p\cong\mathfrak{g}^{reg}_{\bar{\mathbb{F}}_p}$.

Proposition 2.9. *The restrictions of $p_*\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}}(\lambda)$ and $p_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda)$ to $\mathfrak{g}^{reg}$ are locally free.*

Remark 2.2. However, $p_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda)$ is not flat over $\mathfrak{g}$.

Proof. It is clear that $p_*\mathcal{O}_{\mathcal{P}_\lambda\times\mathfrak{g}}(\lambda)$ is free on $\mathfrak{g}$. Therefore, we concentrate on $p_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda)$. It is enough to prove that $p_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda\otimes\bar{\mathbb{F}}_p}(\lambda)$ is locally free over $\mathfrak{g}^{reg}_{\bar{\mathbb{F}}_p}$ of the same rank. Therefore, we base change everything to $\bar{\mathbb{F}}_p$ without changing the notation, where p is a very good prime of G.

Let $L_\lambda\subset P_\lambda\subset G$ be the standard Levi subgroup of P_λ. Then the Weyl group of L_λ is W_λ, the stablizer of λ in W. We construct a map $\tilde{\mathfrak{g}}_\lambda\to\mathfrak{t}/\!\!/W_\lambda = \mathrm{Spec}(S\mathfrak{t}^*)^{W_\lambda}$. Namely, let U_λ be the unipotent radical of P_α and $\mathfrak{n}_\lambda$ be the nilpotent radical of $\mathfrak{p}_\lambda$. Then $P_\lambda/U_\lambda\cong L_\lambda$ and $\mathfrak{p}_\lambda/\mathfrak{n}_\lambda\cong\mathfrak{l}_\lambda=\mathrm{Lie}L_\lambda$. We thus obtain the following map

$$G\times^{P_\lambda}\mathfrak{p}_\lambda\to G\times^{P_\lambda}(\mathfrak{p}_\lambda/\mathfrak{n}_\lambda)\cong(G/U_\lambda)\times^{L_\lambda}\mathfrak{l}_\lambda\to[\mathfrak{l}_\lambda/L_\lambda]\to\mathfrak{t}/\!\!/W_\lambda, \tag{2.6}$$

where $[\mathfrak{l}_\lambda/L_\lambda]$ is the stack quotient, which maps naturally to $\mathfrak{l}_\lambda/\!\!/L_\lambda=\mathrm{Spec}(S\mathfrak{l}_\lambda^*)^{L_\lambda}$. To justify the morphism $[\mathfrak{l}_\lambda/L_\lambda]\to\mathfrak{t}/\!\!/W_\lambda$, we need to show that the natural map $\mathfrak{t}/\!\!/W_\lambda\to\mathfrak{l}_\lambda/\!\!/L_\lambda$ is an isomorphism.

This fact may be well known to experts. However, since we cannot locate a reference, we include a proof here. It is well known that $\mathcal{O}_{L_\lambda}^{L_\lambda}\cong\mathcal{O}_T^{W_\lambda}$. Since the characteristic is very good, according to [BR, 9.3.2-9.3.3], there is a G-equivariant morphism $\varphi: G\to\mathfrak{g}$ sending the unit $e\in G$ to the origin $0\in\mathfrak{g}$ and $(D\varphi)_e=\mathrm{id}$. One can argument as in Lemma 2.7 that it must map L_λ to $\mathfrak{l}_\lambda$ in an L_λ-equivariant way and $T\to\mathfrak{t}$ in a W-equivariant way. We thus obtain the following commutative diagram

$$\begin{array}{ccc} T/\!\!/W_\lambda & \xrightarrow{\cong} & L_\lambda/\!\!/L_\lambda \\ \downarrow & & \downarrow \\ \mathfrak{t}/\!\!/W_\lambda & \longrightarrow & \mathfrak{l}_\lambda/\!\!/L_\lambda \end{array}$$

By Theorem 4.1 of *loc. cit.*, the vertical morphisms in the above diagram are étale around the unit e. Therefore, $\mathfrak{t}/\!\!/W_\lambda\to\mathfrak{l}_\lambda/\!\!/L_\lambda$ is étale around the origin 0. Observe that both $(S\mathfrak{t}^*)^{W_\lambda}$ and $(S\mathfrak{l}_\lambda^*)^{L_\lambda}$ are positive graded and the map respects the grading. Therefore, $(S\mathfrak{l}_\lambda^*)^{L_\lambda}\cong(S\mathfrak{t}^*)^{W_\lambda}$.

We come back to the proof of the proposition. It is easy to see that the map (2.6) gives rise to the following commutative diagram

$$\begin{tikzcd} \tilde{\mathfrak{g}}_\lambda \arrow[r] \arrow[d, "p"'] & \mathfrak{t}/\!\!/W_\lambda \arrow[d] \\ \mathfrak{g} \arrow[r, "\chi"] & \mathfrak{t}/\!\!/W, \end{tikzcd} \tag{2.7}$$

where $\chi : \mathfrak{g} \to \mathfrak{t}/\!\!/W$ is the Chevalley map as in Proposition 3.1. Now, the proposition is the consequence of the following two facts (i) the projection $\mathfrak{t}/\!\!/W_\lambda \to \mathfrak{t}/\!\!/W$ is faithfully flat over $\mathbb{Z}_S$; and (ii) over $\mathfrak{g}^{reg}$, the above diagram is Cartesian.

To see (i), we use the result of [De] that the projections $\mathfrak{t}/\!\!/W_\lambda$ and $\mathfrak{t} \to \mathfrak{t}/\!\!/W$ are faithfully flat. Therefore, $\mathfrak{t}/\!\!/W_\lambda \to \mathfrak{t}/\!\!/W$ is also faithfully flat. To see (ii), let $\tilde{\mathfrak{g}}_\lambda^{reg} = p^{-1}(\mathfrak{g}^{reg})$. We want to show that

$$\tilde{\mathfrak{g}}_\lambda^{reg} \to \mathfrak{g}^{reg} \times_{\mathfrak{t}/\!\!/W} \mathfrak{t}/\!\!/W_\lambda$$

is an isomorphism. Since it is proper and quasi-finite, it is finite. Since both schemes are smooth[3], it is flat. Finally, the assertion follows from the fact that this map is an isomorphism over $\mathfrak{g}^{reg,ss} \times_{\mathfrak{t}/\!\!/W} \mathfrak{t}/\!\!/W_\lambda$, where $\mathfrak{g}^{reg,ss}$ is the open subscheme of $\mathfrak{g}^{reg}$ consisting of regular semi-simple elements. □

An easy application of the above proof is

Proposition 2.10. *The surjective homomorphism*

$$\Gamma(\mathcal{P}_\lambda \times \mathfrak{g}, \mathcal{O}_{\mathcal{P}_\lambda \times \mathfrak{g}}(\lambda)) \twoheadrightarrow \Gamma(\tilde{\mathfrak{g}}_\lambda, \mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda)) \tag{2.8}$$

is an isomorphism if and only if λ is zero or a minuscule weight of G w.r.t B.

Proof. If $\lambda = 0$, everything is clear. So we assume that λ is not zero in the following. If λ is minuscule, then the morphism of $\mathcal{O}_{\mathfrak{g}}$-modules (2.8) is an isomorphism over the generic point of $\mathfrak{g}$ by the following reason. Since the diagram (2.7) is Cartensian over $\mathfrak{g}^{reg}$ (one can see easily that the diagram (2.7) descends to $\mathbb{Z}_S$), over the generic point of $\mathfrak{g}$, $p_*\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda))$ has rank $|W/W_\lambda|$, which is the same as the generic rank of $p_*\mathcal{O}_{\mathcal{P}_\lambda \times \mathfrak{g}}(\lambda)$ since λ is minuscule. Therefore, the kernel of (2.8) is a torsion module over $\mathcal{O}_{\mathfrak{g}}$. However, since $p_*\mathcal{O}_{\mathcal{P}_\lambda \times \mathfrak{g}}(\lambda)$ is a free $\mathcal{O}_{\mathfrak{g}}$-module, the kernel must be zero.

Conversely, if (2.8) is an isomorphism. Then the $\mathbb{Z}_S$-rank of $\Gamma(\mathcal{P}_\lambda, \mathcal{O}(\lambda))$, which is the same as the rank of $p_*\mathcal{O}_{\mathcal{P}_\lambda \times \mathfrak{g}}(\lambda)$ as an $\mathcal{O}_{\mathfrak{g}}$-module, is $|W/W_\lambda|$. The can happen only when λ is minuscule. □

This proposition gives another characterization of minuscule weights of $\mathfrak{g}$, which may be generalized to the Kac-Moody algebras.

3. Proof of Theorem 1.1

3.1. **Regular centralizer.** In this subsection, we review the regular centralizer group scheme of G. The regular centralizer group scheme is well-known when G is an algebraic group over a field. We just write down its counterpart for G being a Chevalley group scheme over $\mathbb{Z}_S$.

[3]The smoothness of $\mathfrak{g}^{reg} \times_{\mathfrak{t}/\!\!/W} \mathfrak{t}/\!\!/W_\lambda$ follows from the smoothness of $\chi : \mathfrak{g}^{reg} \to \mathfrak{t}/\!\!/W$, see [Sl, Chap. II, 3.14].

The centralizer group scheme I over $\mathfrak{g}$ by definition is the group scheme that fits into the following Cartesian diagram

$$\begin{tikzcd} I \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d, "\Delta"] \\ G \times \mathfrak{g} \arrow[r, "\mathrm{Ad}\times 1"] & \mathfrak{g} \times \mathfrak{g}. \end{tikzcd}$$

It is easy to see that $\mathfrak{g}^{reg}$ as defined in §2.4 is the open subscheme of $\mathfrak{g}$ over which the fibers of I have the minimal dimension. It is known that $I|_{\mathfrak{g}^{reg}}$ is commutative. The following proposition is known when the base is a field of very good characteristic (cf. [De, Sl, Ng]). Easy argument will imply that it also hold when the base is $\mathbb{Z}_S$.

Proposition 3.1. (1) *The natural map $\mathcal{O}_{\mathfrak{g}}^G \to \mathcal{O}_{\mathfrak{t}}^W$ is an isomorphism, and they are both isomorphic to a polynomial algebra over $\mathbb{Z}_S$. Denote $\mathfrak{t}/\!\!/W =$ Spec $\mathcal{O}_{\mathfrak{t}}^W$, and let $\varpi : \mathfrak{t} \to \mathfrak{c}$, $\chi : \mathfrak{g} \to \mathfrak{c}$ be the natural projections. Then both ϖ, χ are faithfully flat. In addition, the restriction $\chi|_{\mathfrak{g}^{reg}}$ is smooth.*

(2) *There is a (unique up to isomorphism) smooth commutative group scheme J over $\mathfrak{c}$, such that*

$$\chi^* J|_{\mathfrak{g}^{reg}} \cong I|_{\mathfrak{g}^{reg}}.$$

In literature, J is usually called the regular centralizer group scheme of G.

It is clear from the definition that $[\mathfrak{g}^{reg}/G] \cong \mathbb{B}J$ as stacks over $\mathfrak{c}$. We have the following natural functor from the category of G-equivariant coherent sheaves on $\mathfrak{g}$ to the category of J-modules

$$\mathrm{Coh}([\mathfrak{g}/G]) \to \mathrm{Coh}([\mathfrak{g}^{reg}/G]) \cong \mathrm{Coh}(\mathbb{B}J) \cong J\text{-mod}\,. \tag{3.1}$$

In concrete terms, this means that a G-equivariant coherent sheaf on $\mathfrak{g}^{reg}$ descends to a coherent sheaf on $\mathfrak{c}$ by faithfully flat descent. In addition, such sheaf carry on a natural J-action. Let

$$\mathcal{S}^{-w_0(\lambda)} \to \mathcal{L}_*^{-w_0(\lambda)} \tag{3.2}$$

be the J-module morphism that is the image of (1.3) under this functor. By Proposition 2.9, as coherent sheaves on $\mathfrak{c}$, both $\mathcal{S}^{-w_0(\lambda)}$ and $\mathcal{L}_*^{-w_0(\lambda)}$ are locally free. Let us denote J-module map dual to (3.2) by

$$\mathcal{L}_!^{\lambda} \to \mathcal{W}^{\lambda}. \tag{3.3}$$

The following lemma is a consequence of the fact that there is a G-module morphism $W^\lambda \to S^\lambda$, where $W^\lambda = \Gamma(\mathcal{P}_\lambda, \mathcal{O}(\lambda))^*$ is the Weyl module and $S^\lambda = \Gamma(\mathcal{P}_{-w_0(\lambda)}, \mathcal{O}(-w_0(\lambda))$ is the Schur module.

Lemma 3.2. *There is a natural J-module morphism $\mathcal{W}^\lambda \to \mathcal{S}^\lambda$.*

3.2. **Review of the equivariant homology of the affine Grassmannian.** Let $\mathrm{Gr} = G^\vee(F)/G^\vee(\mathcal{O})$ be the affine Grassmannian of $G^\vee$. This is a union of projective varieties. In [YZ], the (equivariant) homology of Gr with $\mathbb{Z}_S$-coefficients is expressed as the algebraic functions on certain group schemes associated to G. Let us briefly recall it.

First, let f be the unique W-invariant quadratic form on $\mathfrak{t}$ that takes 2 on long coroots. Then f gives rise to a W-equivariant isomorphism $f : \mathfrak{t}^* \cong \mathfrak{t}$ over $\mathbb{Z}_S$. Therefore, there is an isomorphism

$$\mathrm{Spec} H^*(\mathbb{B}G^\vee) \cong \mathfrak{t}^*/\!\!/W \cong \mathfrak{t}/\!\!/W \tag{3.4}$$

It is not hard to show that $\mathrm{Spec} H_*^{G^\vee(\mathcal{O})}(\mathrm{Gr})$ is a commutative group scheme over $\mathfrak{t}/\!\!/W$.

Proposition 3.3. *(See* [YZ, Proposition 6.6]*.)There is a canonical isomorphism*

$$\mathrm{Spec}\ H_*^{G^\vee(\mathcal{O})}(\mathrm{Gr}) \cong J.$$

Now if $\mathcal{F} \in D_{G^\vee(\mathcal{O})}(\mathrm{Gr})$, then $H^*_{G^\vee(\mathcal{O})}(\mathrm{Gr}, \mathcal{F})$ is a comodule over $H_*^{G^\vee(\mathcal{O})}(\mathrm{Gr})$, and therefore a module over J.

Let $T^\vee$ be the maximal torus of $G^\vee$. We should also review the $T^\vee$-equivariant homology of Gr. According to [YZ], the $T^\vee$-equivariant Chern class of the determinantal line bundle of Gr gives rise to a map $e^T : \mathrm{Spec}\ H(\mathbb{B}T^\vee) \cong \mathfrak{t}^* \to \mathfrak{g}^{reg}$ making the following diagram commute

$$\begin{tikzcd} \mathfrak{t}^* \arrow[r, "e^T"] \arrow[d, "f"'] & \mathfrak{g}^{reg} \arrow[d, "\chi"] \\ \mathfrak{t} \arrow[r, "\varpi"] & \mathfrak{c} \end{tikzcd} \tag{3.5}$$

Proposition 3.4. *(See* [YZ, Theorem 6.1]*.) There is a canonical isomorphism of group schemes*

$$\mathrm{Spec} H_*^{T^\vee}(\mathrm{Gr}) \cong (e^T)^* I.$$

3.3. **Proof of Theorem 1.1.** We begin with

Proposition 3.5. *The natural map*

$$H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda) \to H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i_*^\lambda) \tag{3.6}$$

is a surjective map of free $H(\mathbb{B}G^\vee)$-modules. Dually, the natural map

$$H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i_!^\lambda) \to H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_!^\lambda) \tag{3.7}$$

is splitting injective as free $H(\mathbb{B}G^\vee)$-modules.

Proof. We first show that $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda)$ is finitely generated and flat over $H(\mathbb{B}G^\vee)$. Then since $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda)$ is graded, it must be free.

Since $H(\mathbb{B}G^\vee) \to H(\mathbb{B}T^\vee)$ is finite and faithfully flat by taking $\mathbb{Z}_S$-coefficients (see Proposition 3.1), it is enough to show $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda) \otimes_{H(\mathbb{B}G^\vee)} H(\mathbb{B}T^\vee)$ is finitely generated and flat over $H(\mathbb{B}T^\vee)$. First, the natural map

$$H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda) \otimes_{H(\mathbb{B}G^\vee)} H(\mathbb{B}T^\vee) \to H_{T^\vee}(\mathrm{Gr}, I_*^\lambda)$$

is an isomorphism and by [YZ, Lemma 2.2], there is a canonical isomorphism of $H(\mathbb{B}T^\vee)$-modules

$$H_{T^\vee}(\mathrm{Gr}, I_*^\lambda) \cong H(\mathrm{Gr}, I_*^\lambda) \otimes H(\mathbb{B}T^\vee).$$

Secondly, according to [MV, §3], $H(\mathrm{Gr}, I_*^\lambda)$ is a free $\mathbb{Z}_S$-module of finite rank. Therefore, $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda) \otimes_{H(\mathbb{B}G^\vee)} H(\mathbb{B}T^\vee)$ is finitely generated and flat over $H(\mathbb{B}T^\vee)$. Similarly, one can show that $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i_*^\lambda)$, $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i_!^\lambda)$ and $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_!^\lambda)$ are also free over $H(\mathbb{B}G^\vee)$.

Let $\mathbb{D}$ be the Verdier duality functor. It is known by [MV, Proposition 8.1] that $\mathbb{D}I_*^\lambda = I_!^\lambda$. Therefore, (3.6) implies (3.7). In addition, the Leray spectral sequence $H(\mathbb{B}G^\vee, H(\mathrm{Gr}, I_*^\lambda)) \Rightarrow H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda)$ (resp. $H(\mathbb{B}G^\vee, H(\mathrm{Gr}, i_*^\lambda)) \Rightarrow H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i_*^\lambda)$)

degenerates since all the cohomology concentrate on even degrees. Therefore, it is enough to show that

$$H(\mathrm{Gr}, I_*^\lambda) \to H(\mathrm{Gr}, i_*^\lambda)$$

is surjective.

Observe that the natural map $\mathbb{Z}_S[\dim \overline{\mathrm{Gr}}^\lambda] \to i_*^\lambda$ of complex of sheaves on $\overline{\mathrm{Gr}}^\lambda$ factors as

$$\mathbb{Z}_S[\dim \overline{\mathrm{Gr}}^\lambda] \to I_*^\lambda \to i_*^\lambda.$$

Now the proposition follows from the fact that $H(\overline{\mathrm{Gr}}^\lambda) \to H(\mathrm{Gr}^\lambda)$ is surjective. □

Remark 3.1. This remark will not be used in the sequel. All sheaves in the remark are taken $\mathbb{Z}$-coefficients. Observe that $H^{-\dim \overline{\mathrm{Gr}}^\lambda}(\mathrm{Gr}, I_!^\lambda) \cong \mathbb{Z}$ and its unique (up to sign) basis induces $\mathbb{Z}[\dim \overline{\mathrm{Gr}}^\lambda] \to I_!^\lambda$. One can show that the natural map $i_!^\lambda \to \mathbb{Z}[\dim \overline{\mathrm{Gr}}^\lambda]$ followed by this $\mathbb{Z}[\dim \overline{\mathrm{Gr}}^\lambda] \to I_!^\lambda$ is the natural perverse truncation map (up to sign). In other words, we have

$$i_!^\lambda \to \mathbb{Z}[\dim \overline{\mathrm{Gr}}^\lambda] \to I_!^\lambda.$$

The proof of the proposition implies that $H(\mathrm{Gr}, i_!^\lambda) \to H(\mathrm{Gr}, I_!^\lambda)$ is splitting injective (as $\mathbb{Z}$-modules). But one can show a stronger result holds. Namely, the map $H(\overline{\mathrm{Gr}}^\lambda, \mathbb{Z}[\dim \overline{\mathrm{Gr}}^\lambda]) \to H(\mathrm{Gr}, I_!^\lambda)$ is splitting injective.

Now we begin to prove Theorem 1.1. We first show that there is a commutative diagram of J-modules

$$\begin{tikzcd} H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_!^\lambda) \arrow[r] \arrow[d, "\cong"'] & H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda) \arrow[d, "\cong"] \\ \mathcal{W}^\lambda \arrow[r] & \mathcal{S}^\lambda \end{tikzcd}.$$

It is enough to show that after the base change $\varpi \circ f : \mathfrak{t} \to \mathfrak{c}$, such a diagram exists as $(\varpi \circ f)^* J \cong (e^T)^* I$-modules (see (3.5)). On the one hand, by definition, $(\varpi \circ f)^* \mathcal{W}^\lambda \to (\varpi \circ f)^* \mathcal{S}^\lambda$ as the $(e^T)^* I$-modules is the same as $\mathcal{O}_{\mathfrak{t}^*} \otimes W^\lambda \to \mathcal{O}_{\mathfrak{t}^*} \otimes S^\lambda$. On the other hand, $(\varpi \circ f)^* H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_!^\lambda) \to (\varpi \circ f)^* H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda)$ as $(e^T)^* I \cong \mathrm{Spec}\, H_*^{T^\vee}(\mathrm{Gr})$-modules, is isomorphic to $H_{T^\vee}(\mathrm{Gr}, I_!^\lambda) \to H_{T^\vee}(\mathrm{Gr}, I_*^\lambda)$, which is also isomorphic to $\mathcal{O}_{\mathfrak{t}^*} \otimes W^\lambda \to \mathcal{O}_{\mathfrak{t}^*} \otimes S^\lambda$ by [YZ].

To finish the proof of the theorem, we will show that there exists a J-modules isomorphism $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i_*^\lambda) \to \mathcal{L}_*^\lambda$ making the following diagram commute

$$\begin{tikzcd} H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda) \arrow[r] \arrow[d, "\cong"] & H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i_*^\lambda) \arrow[d, "\cong"] \\ \mathcal{S}^\lambda \arrow[r] & \mathcal{L}_*^\lambda. \end{tikzcd}$$

The isomorphism $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i_!^\lambda) \to \mathcal{L}_!^\lambda$ is deduced similarly.

All the modules in the above diagram are (locally) free over $\mathfrak{t}/\!\!/W$ (see Proposition 2.9 and Proposition 3.5). In addition, the horizontal maps are surjective by Proposition 3.5 and Theorem 1.3. Therefore, to show that there is a map $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i_*^\lambda) \to \mathcal{L}_*^\lambda$ making the above diagram commute, it is enough to show there is such a map at the generic point $\mathfrak{t}/\!\!/W$. We can even just prove that such a map exists over ξ, where ξ is the generic point of $\mathfrak{t}$, which maps to $\mathfrak{t}/\!\!/W$ via $\xi \to \mathfrak{t} \xrightarrow{\varpi} \mathfrak{t}/\!\!/W$.

We know that J_ξ is isomorphic to $I_\xi = Z_{G_\xi}(\xi)$, the centralizer of ξ in G_ξ. Since $\xi \in \mathfrak{t}$, $Z_\xi(G_\xi) \cong T_\xi$. By the equivariant localization theorem, the map $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda)_\xi \to H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i_*^\lambda)_\xi$ as modules over $\mathrm{Spec} H_*^{G^\vee(\mathcal{O})}(\mathrm{Gr})_\xi \cong T_\xi$ is killing all the weight spaces of $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, I_*^\lambda)_\xi$ whose weights are not in the W-orbit of λ. On the other hand, the base change of (1.2) to ξ is the same as the closed embedding of the T_ξ-fixed point subscheme $(\mathcal{P}_\lambda)_\xi$ of $\mathcal{P}_\lambda$ to $\mathcal{P}_\lambda$. Therefore, by the localization theorem of coherent sheaves, the map $(\mathcal{S}^\lambda)_\xi \to (\mathcal{L}_*^\lambda)_\xi$ as T_ξ-modules also corresponds to killing all the weight spaces of $(\mathcal{S}^\lambda)_\xi$ whose weights are not in the W-orbit of λ. It is clear from the above descriptions that a map $H_{G^\vee(\mathcal{O})}(\mathrm{Gr}, i_*^\lambda)_\xi \to (\mathcal{L}_*^\lambda)_\xi$ exists.

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